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C. Pellegrini: NON LINEAR EFFECTS ON THE DAMPING CONSTANTS OF ELECTRON OSCILLATIONS IN A SYNCHROTRON.

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C. Pellegrini: NON LINEAR EFFECTS ON THE DAMPING CONSTANTS OF ELECTRON OSCILLATIONS IN A SYNCHROTRON.

1) In a previous paper*[(1)] the effects of radiation on electron oscillations were evaluated for a class of circular accelerators satisfying the condition that all non linear terms can be neglected.

Recently it has been shown by Hereward[(2)] that the non linearity of the expression for the radiated energy can be important in strong-focusing machines and that it can be used to avoid antidamping on the radial betatron oscillation mode.

In view of this fact it seems useful to extend our previous calculations so that can be applied to this case also.

Since a complete treatment of the problem is much too difficult, we will essentially follow Hereward and make the approximation that the zero order equations, i.e. those in which radiation and the radio-frequency cavities (R.F.) are neglected, are linear while non linear terms appear in the expression for the radiated energy, w .

These non linear terms will modify the damping constants only if the reference trajectory is displaced or, what is same, if the frequency of the R.F. is shifted.

This second point of view will be taken here.

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* In what follows this paper will be mentioned as I. We will also use the same notations of I.

2) The equations of motion are

$$(1)\qquad \begin{cases} \sigma' = Kx \\ x'' + (K^2 + A)x = -Kp \\ z'' - Az = 0 \end{cases}$$

where

$$A = -K^2 n$$

in a magnet and

$$A = \frac{G}{H\rho}$$

in a quadrupole.

Their solution is[3]

$$(2)\qquad \begin{cases} x = x_o\sqrt{\beta_r}\ \cos\ \gamma_r - p\psi = x_\beta - p\psi \\ z = z_o\sqrt{\beta_v}\ \cos\ \gamma_v = z_\beta \end{cases}$$

The expression for w is

$$(3)\qquad w = \frac{2}{3}\, r_e\, \gamma^3 (1+p)^2 \left\{ (Ax-K)^2 + A^2 z^2 \right\}$$

or, neglecting terms of third or fourth order in x, z or p:

$$(4)\qquad w = \frac{2}{3}\, r_e\, \gamma^3 \left\{ K^2 - 2AKx + A^2(x^2+z^2) + 2K^2 p + 4AKxp + K^2p^2 \right\}$$

Of three quadratic terms only that proportional to A^2, i.e. to the square of the magnetic field gradient, can be important in practical cases, so that from now on only this one will be considered.

Substituting (2) in (4) we obtain

$$(5)\qquad \begin{aligned} w = {} & \frac{2}{3}\, r_e\, \gamma^3 \left\{ K^2(1+2p) + 2AKp\psi + A^2\psi^2 p^2 \right\} + \\ & + \frac{2}{3}\, r_e\, \gamma^3 \left\{ -2AKx_\beta + A^2(x_\beta^2 + z_\beta)^2 - 2A^2\psi\, x_\beta\, p \right\} \end{aligned}$$

Now we assume that the R.F. gives to the particles the energy $\mathcal{E}_r$ lost on the closed orbit corresponding to the energy $E_s(1+\bar{p})$, i.e. that the synchronous particle is that with energy $E_s(1+\bar{p})$.

Writing

$$(6) \qquad \mathcal{E}_r = \frac{eV_o}{E_s(1+\bar{p})} \cos\varphi = \frac{eV_o}{E_s(1+\bar{p})} \cos\varphi_s + f^2 \sigma^{*}(1C_s)$$

our assumption is that

$$(7) \qquad \frac{eV_o}{E_s(1+\bar{p})} \cos\varphi_s = C_s \left\langle \frac{2}{3} r_e \gamma^3 \left[K^2(1+2\bar{p}) + 2AK\psi\bar{p} + A^2\psi^2 p^2 \right] \right\rangle = C_s \langle \bar{w}_s \rangle ,$$

while usually

$$\frac{eV_o}{E_s} \cos\varphi_s = C_s \left\langle \frac{2}{3} r_e \gamma^3 K^2 \right\rangle = C_s \langle w_s \rangle .$$

Hence σ^{*} differs from σ since it must measure the shift with respect to the new syncronous particle; this implies that, neglecting the betatron oscillations,

$$(8) \qquad \sigma^{*} = - K(p - \bar{p})\psi .$$

3) Let us start by evaluating the synchrotron damping.We neglect the terms giving rise to the equilibrium dimensions of the beam since they are left unchanged.

Eq. (23) of I is now substituted by

$$D^2(p - \bar{p}) + \nu^2(p - \bar{p}) = 0 ;$$

the solution is:

$$p - \bar{p} = p_o \cos \gamma_s$$

where

$$\gamma_s = s\,\nu_s + \alpha_s .$$

Introducing the notation

$$\chi = \frac{2}{3} r_e \gamma^3$$

equation (45) of I becomes

$$\Delta p = \left\{ \left\langle \chi \left[K^2(1+2\bar{p}) + 2KA\psi\bar{p} + A^2\psi^2\bar{p}^2 \right] \right\rangle - \chi \left[K^2(1+2p) + 2AK\psi p + A^2\psi^2 p^2 \right] \right\} \Delta s\, \delta(s-s_i)$$

Now

$$\Delta(p_o)^2 = 2\Delta p(p - \bar{p})$$

and averaging we obtain

$$D(p_o)^2 = -p_o^2 \left\langle \chi \left[2K^2 + 2AK\psi + 2A^2\psi^2 p \right] \right\rangle$$

so that

$$(9) \qquad \frac{1}{\tau_s} = -\tfrac{1}{2} \left\langle \chi \left(2K^2 + 2AK\psi + 2A^2\psi^2 \bar{p} \right) \right\rangle = -\tfrac{1}{2} \left\langle \frac{dw}{dp} \right\rangle$$

The average must be performed on the closed orbit corresponding to the energy $E_s(1+p)$ and neglecting betatron oscillations.

The betatron damping, evaluated with the same procedure used in I, is

$$(10) \qquad \begin{aligned} \frac{1}{\tau_{\beta r}} &= +\tfrac{1}{2} \left\langle \chi \left(2AK\psi + 2A^2\psi^2 \bar{p} \right) \right\rangle - \tfrac{1}{2} \langle \bar{w}_s \rangle = \\ &= \tfrac{1}{2} \left\langle \frac{dw}{dp} \right\rangle - \frac{3}{2} \langle \bar{w}_s \rangle \end{aligned}$$

while for the vertical damping we obtain

$$\frac{1}{\tau_{\beta v}} = -\tfrac{1}{2} \langle \bar{w}_s \rangle \ .$$

In writing (10), (11) we have neglected the small differences between the values of $\langle w \rangle$ on different closed orbits. It is interesting to note that we have

$$\frac{1}{\tau_s} + \frac{1}{\tau_{\beta r}} + \frac{1}{\tau_{\beta v}} = -2 \langle \bar{w}_s \rangle \ .$$

As we already said the results obtained are valid only if the R.F. frequency is shifted, otherwise the non linear terms would give no contribution.

In fact, setting $\bar{p} = 0$ in (7), we obtain

$$\frac{1}{\tau_s} = -\tfrac{1}{2} \left\langle \chi \left(2K^2 + 2AK\psi \right) \right\rangle$$

$$\frac{1}{\tau_{\beta v}} = -\tfrac{1}{2} \langle w_s \rangle$$

$$\frac{1}{\tau_{\beta r}} = \tfrac{1}{2} \left\langle \chi \left(2AK\psi + 2A^2\psi^2 p \right) \right\rangle - \tfrac{1}{2} \langle w_s \rangle \ .$$

Clearly the only term in which contributions of the non linearity appear is $1/\tau_{\beta r}$. However, due to the fact that the synchrotron and betatron modes are uncoupled, we can average $1/\tau_{\beta r}$ over a synchrotron period so that the term $A^2\psi^2 p$ disappear.

For an isomagnetic machine and assuming

$$A = -K^2 n, \quad \psi = R/\nu^2, \quad n = \xi N^2 = \xi\,(\text{number of magnets})^2, \quad \alpha = 1/\nu^2,$$

the synchrotron damping constant (9) becomes

$$\frac{1}{\tau_s} = -\tfrac{1}{2}\langle w_s\rangle \left(4 - \alpha + 2\xi^2 N^4 \alpha^2 \bar{p}\right)$$

and this formula is in agreement with that of Hereward. The coefficient ξ is in general a function of ν.

In the case of a separated functions machine, with weak-focusing bending magnets, we have

$$\frac{1}{\tau_s} \simeq -\tfrac{1}{2}\langle w_s\rangle \left(\frac{3-4n}{1-n} + 2\left(\frac{G}{H}\right)^2 \frac{R^2}{\nu^4} \frac{\lambda_Q}{\lambda_M}\bar{p}\right)$$

where G is quadrupole gradient, H the guide magnetic field and λ_Q/λ_M is the ratio between the quadrupoles length. and the magnet length.

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(1) - C. Pellegrini, Suppl. Nuovo Cimento 22, 603 (1961)

(2) - H.G. Hereward, International Conference on High Energy Accelerators, Brookhaven 1961, pag. 222

(3) - E.D. Courant and H.S. Snyder, Ann. Phys. 3, 1 (1958)